Ideas for a radical reform of mathematics education, après Arnold, Gelfand, and Vavilov

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A few words on Nikolai Vavilov

- Kolya was one of the people who directly influenced my views of mathematics and mathematics education.
- I many times taught different versions of a course **Concrete Group Theory** (the name of his book).
- For example, I explained to undergraduate students that the cross product of vectors in 3d Euclidean space is the Lie algebra of the Euclidean rotation group $\mathrm{SO}_3(\mathbb{R})$—a fundamental fact seldom mentioned in university courses. ↓

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- In the 1980s, a few conversations with Kolya about linguistics were a revelation for me. later helped me to teach students from countries from A to Z, literally: from Afghanistan to Zambia.

One point of disagreement with Nikolai

I disagree with Kolya's assessment of the role of formal proofs and the use interactive proof assistance and automated proof checkers.

- For Kolya, proof is a form of communication between **humans**, with all the consequences of that.
- But humans need formal, proof theory-based, proofs for being confident in communication with **computers**.
- Kolya loved quotes; I'll start with this one:

The most important issue of the 21st Century is the relationship between people and machines.

Whitfield Diffie

Agreement / Disagreement

A very good analysis of Nikolai Vavilov's approach to mathematics education was provided by Alexei Semenov in his talk.

- I accept some Kolya's ideas wholeheartedly, disagree with others.
- The reason for disagreement is the most mild: I simply look at mathematics education from a different point of view:

This is economy, stupid.
(Falsely attributed to Bill Clinton)

Economy and politics

- I taught in 4 different countries with 4 different education systems.
- Everywhere, mathematics education discourse is politicised.
- In Britain, it is politicised to unbelievable degree and underlying reasons could be seen with much better clarity.
- As a rule, this kind of polarisation of views in politics is a manifestation of a tectonic rift in the economic foundations.

What's It Really All About?

> *But Didactylos posed the famous philosophical conundrum: "Yes, But What's It* **Really** *All About, Then, When You Get Right Down To It, I* **Mean** *Really?"*
>
> *Terry Pratchett*

- Because of changing patterns of division of labour in the economy, the **Mainstream** mathematics education **for everyone** is economically redundant and unsustainable.
- Not surprising it is **decaying** and **dying**.
- I do not know what to do with it.
- One thing is clear: we should not spread panic.

Band on the deck of *Titanic*

Most policies in mathematics education can be divided in two categories:

- rearranging chairs on the deck of *Titanic* (the preferred option of the political Right);
- helping disadvantaged passengers to get better chairs on the deck of *Titanic* (the preferred option of the political Left).

We should stay in the famous band that continues to play regardless.

For the *Titanic* band, this was their last call of duty: prevent panic, help the orderly evacuation.

But how to save at least someone?

The society needs a **certain number** of properly mathematically educated people

Why?

- *Because mathematicians are* **stem cells** *on of a technologically advanced society: in the hour of real need, they can morph into anything.*
- *And because*

> *Mathematics: Solving tomorrow's problems yesterday.*
>
> *Rob Wilson*

Mathematicians have access to the wisdom of mathematics accumulated over millennia.

Inevitability of a split

We have to accept the inevitability of maths education splitting, in

- **Mainstream** – for the vast majority of population.
- "**Deep Stream**", with a further subdivision at some point, into substreams
  - for future mathematicians and computers scientists
  - for non-mathematicians who will be professionally using mathematics.

Surprisingly, there is no coherent discussion of these issues.

Will come to discussion of Deep Stream a bit later, but first – a few words on Mainstream.

Can the Mainstream maths education be saved?

I do not thinks so. Why?

There was no money

THERE IS NO MONEY

THERE WILL BE NO MONEY

Maybe it can be done on the cheap?

Educators hoping that entrusting teaching to computers could help to save money are turkeys voting for Christmas:

- Any skills which can be taught by computers are economically redundant: computer will e able to perform the relevant tasks more efficeintly and cheaper than humans.
- Flight simulators developed for training pilots now can fly planes better than humans. Only legislative inertia prevents them from doing so.
- It is unlikely that we will ever have legislation preventing computers from performing mathematical tasks.

Protecting the mathematical community

We should attend to Mainstream education as to a gravely ill patient,
but
our first priority should be saving the professional mathematical community.
If not us, who else will do that?

Timescale

The cycle of reproduction of mathematics

primary school – high school – college/university/ – return to school as a teacher

is at least 15 years; if it continues as

university/ – PhD studies – postdoc – return to university as a lecturer,

it is at least 20 years.

Any serious reform needs to be planned and implemented on this timescale.

Reverse analysis

How to nurture and educate people who will be able to create and use the XXI century mathematics?

- We cannot just wait that they will just somehow appear.
- Let us analyse the situation from the aim to the current situation, in reverse steps:
  - PhD study
  - MSc
  - BSc
  - High school
  - Middle school
  - Primary school

There are obvious gaps, insufficient preparation, between all these steps. In England, the situation is especially bad.

Where will the next generation of UK mathematicians will come from?

In 2005, Tony Gardiner and I organised a one-day meeting to discuss this question.

The answer was: from overseas.

Since then, the situation has not changed.

My last PhD students were from Croatia, Finland, and Hungary.

Deep Stream

Filling gaps should start from pre-school and primary school, which leads to the concept of **Deep Stream Mathematics Education**:

> *Mathematics education in which every stage, starting from pre-school, is designed to fit the individual cognitive profile of the child and to serve as propaedeutics of his/her even deeper study of mathematics at later stages of education—including transition to higher level of abstraction and changes of conceptual frameworks.*

Alas, in most countries this approach cannot be implemented in the mainstream schools.

To succeed in the Deep Stream, a child needs a strong support from his/her family.

The aims of the Deep Stream

Developing in children skills of

- *identification of hidden structures in the real world, in IT, in mathematics*
- *abstraction*
- *reification and encapsulation / de-encapsulation*
- *confident handling and creation of proofs*
- *confident and fluent computer programming skills*
- *ability to create, maintain, and use mental images of big complex hierarchically built structures*

Deep Stream: psychological and cognitive traits to nurture

- *ability to engage the subconscious when doing mathematics;*
- *ability to share intuition;*
- *ability to learn by absorption;*
- *ability to compress mathematical knowledge;*
- *capacity for abstract thinking;*
- *being in control of their mathematics.*

Borovik 2017: *Mathematics for makers and mathematics for users*, http://bit.ly/2qYHtst.

Loose network

Deep Stream means

- running a network of mathematical circles, olympiads, Sunday, Winter, Easter, Summer Schools, etc.,
- with the aim of identifying children who must be saved from their schools and directed to mathematical classes, schools, boarding schools of the *Deep Stream*.

This is a colossal task, and it cannot be done without an active participation of the professional mathematical community.

> **Then saith he unto his disciples, The harvest truly is plentious, but the labourers are few.**
>
> *Matthew 9:37 KJV*

This could be quite a challenge

Pavel Alexandrov, the chairman of the organizing committee for the first Moscow Mathematics Olympiad of 1935, wrote in 1936:

> *The Olympiad is the first entry of future mathematicians into the mathematical arena. It should help us* **to select** *these future mathematicians from among our youth, it should help us* **to provide opportunities** *for their further mathematical development and education.*

This was published in Stalin's times when *selection* and *promotion* of the *cadre* was the absolute monopoly of the Communist Party. It was quite daring on part of Alexandrov.

The principal political obstacle

The first Mathematical Olympiads were organized in 1934 in Leningrad by Boris Delaunay.

The aim was to circumvent political restrictions and admit to the University children of people deprived of citizens' rights:

> *army officers and civil servants of the previous tsarist regime, clergy, nobility, capitalists, etc.*

In Britain, the principal obstacle will be the **Diversity, Equity, and Inclusion (DEI)** agenda.

A quote from Dostoevsky

Are you a creature trembling or have the RIGHT?

Fedor Dostoevsky

Selection is paramount

> **Teaching mathematics to the would-be mathematicians is infinitely easier than teaching mathematics to non-mathematicians.**
>
> *Vladimir Rokhlin*

But where to get them, would-be mathematicians?

For that reason, nurturing and selection should go hand-in-hand and criteria should be formulated in terms of **mathematical and cognitive skills**, not curriculum items.

Proto-curriculum

With the organisation administrative structure of the Deep Stream network being so loose, some shared understanding of its purpose and possible curricula is paramount.

What I suggest is a fragment of this curriculum, a ‘course’ which runs from the primary school to the age of 16:
It includes some geometry.

Other parts of mathematics:

- mechanics
- combinatorics
- probability and statistics

need a separate discussion and are not touched here.

Interactions with physics deserve most serious attention.

Arithmetic + Algorithmic + Algebra

Should start in primary school as soon as kids can read and write/type. Its first principle:

> *Learner's answer to an arithmetic or algebraic problem (including mathematical problems described as 'real life' problems) should be an algorithm (and later—executable computer code developed, almost in its entirety, by Learner)—which*
>
> - *solves all problems of the same type;*
> - *helps to check, analyse, and generalise the solution.*

Questions procedure for ‘word problems’

Igor Arnold, 1947:

1. *Teaching arithmetic involves, as a key component, the development of an ability to negotiate situations whose concrete natures represent very different relations between magnitudes and quantities.’*

This means mapping the structures and relations of the real world into operations and relations of arithmetic.

2. *The difference between the “arithmetic” approach to solving problems and the algebraic one is, primarily the need to make a concrete and sensible interpretation of all the values which are used and/or which appear at any stage of the discourse.*

Word problems

The aim is to develop

- full use of "functorial" properties of the mapping from the real world to numbers and other mathematical structures
- depth and functionality of thinking within a limited language.

My favourite simile is the *Essential English* method developed in 1930s by West, Palmer, and Faucett for teaching English as a second language, which was aimed at achieving *full language fluency* within a *limited*, but *functional vocabulary*.

Algebra

Algebra should be introduced simultaneously with the switch from verbal algorithms to computer programming.
Should reflect the demands of computer programming, and include, for example,

- Boolean algebra
- elements of number theory
- modular arithmetic
- finite fields.

Requires the development of the bespoke, and very child-friendly, Domain-Specific Language (DSL)

Links with computer science education

- Mathematics and computer science should be treated as a single subject.
- Exposure to code writing from day one.
- Some seminumerical algorithms from volume 2 of Donald Knuth's *The art of computer programming* beg to be included in the algebra lessons.
- Algebra should be taught and learnt in parallel with user-friendly programming language which includes elements of functional programming.

Abstraction

Kramer 2007: [Abstraction means] the process of removing detail to simplify and focus attention [...]:

- the act of withdrawing or removing something, and
- the act or process of leaving out of consideration one or more properties of a complex object so as to attend to others.

[It is also] the process of generalisation to identify the common core or essence [...]:

- the process of formulating general concepts by abstracting common properties of instances, and
- a general concept formed by extracting common features from specific example.

Encapsulation *après* Weller e.a.

To add two functions f and g to obtain a new function $f + g$:

- Treat the two original functions and the resulting function as objects.
- De-encapsulating back to the two underlying processes and coordinate them by thinking about all of the elements x of the domain and all of the individual transformations $f(x)$ and $g(x)$ at one time.
- Obtain, by adding, the new process, which consists of transforming each x to $f(x) + g(x)$.
- Encapsulated this new process to obtain the new function $f + g$.

Encapsulation / de-encapsulation and Khan Academy

Gulnar has an average score of 87 after 6 tests. What does Gulnar need to get on the next test to finish with an average of 78 on all 7 tests?

Hints provided, one after another, by the Khan Academy website:

Hint 1. Since the average score of the first 6 tests is 87, the sum of the scores of the first 6 tests is $6 \times 87 = 522$.

Hint 2. If Gulnar gets a score of x on the 7th test, then the average score on all 7 tests will be:

$$\frac{522 + x}{7}.$$

Hint 3. This average needs to be equal to 78 so:

$$\frac{522 + x}{7} = 78.$$

Hint 4 $x = 24$.

Encapsulation / de-encapsulation and “Questions method”

“Questions” method circa 1950-s:

Question 1. How many points in total did Gulnar get in 6 tests?
Answer: $6 \times 87 = 522$.

Question 2. How many points in total does Gulnar need to get in 7 tests?
Answer: $7 \times 78 = 546$.

Question 3. How many points does Gulnar need to get in the 7th test?
Answer: $546 - 522 = 24$.

These questions are **not** asked by the teacher, they are expected from **students** themselves.

Questions method

Gulnar has an average score of 87 after 6 tests. What does Gulnar need to get on the next test to finish with an average of 78 on all 7 tests? ↓

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The teacher removes the question from the problem:

Gulnar has an average score of 87 after 6 tests. . . . . . .
. . . . . to finish with an average of 78 on all 7 tests.

and asks instead:

What questions could be asked about these data? ↓

Questions method

> *Gulnar has an average score of 87 after 6 tests. What does Gulnar need to get on the next test to finish with an average of 78 on all 7 tests?* ↓

The teacher removes the question from the problem:

> *Gulnar has an average score of 87 after 6 tests. .... to finish with an average of 78 on all 7 tests.*

and asks instead:

What questions could be asked about these data? ↓

Students are expected to **analyse** and, if needed, **de-encapsulate** the data.

Reification

Wikipedia:

Reification is the process by which an abstract idea about a computer program is turned into an explicit data model or other object created in a programming language.

A computable/addressable object—a resource—is created in a system as a proxy for a non computable/addressable object.

Reification: examples

x	The notorious unknown x of elementary algebra.
$i = \sqrt{-1}$	Imaginary root of -1, the basis of complex analysis.

It took centuries for mathematicians to develop these new mathematical entities.

Is it realistic to expect from schoolchildren to do reification as a normal everyday task?

Reification in arithmetic: intermediate parameters

It takes 5 days for a steamboat to get from St Louis to New Orleans, and 7 days to return back. How long will it take for a raft to drift from St Louis to New Orleans? ↓

Reification in arithmetic: intermediate parameters

It takes 5 days for a steamboat to get from St Louis to New Orleans, and 7 days to return back. How long will it take for a raft to drift from St Louis to New Orleans? ↓

Let us **introduce** a new unit of distance, *lieue* and **set** the distance from StL to NO being equal to $5 \times 7 = 35$ lieue.
Speeds of the steamboat:

$$\text{downstream } \frac{35 \text{ lieue}}{5 \text{ day}} = 7\frac{\text{lieue}}{\text{day}}, \text{ upstream } \frac{35 \text{ lieue}}{7 \text{ day}} = 5\frac{\text{lieue}}{\text{day}}.$$

Since the speed of the current gets added to, or subtracted from, the speed of the steamship in still water, the speed of the current is

$$\frac{7 \text{ lieue/day} - 5 \text{ lieue/day}}{2} = 1\frac{\text{lieue}}{\text{day}}$$

and a raft will drift for 35 lieue/1 lieue/day $=$ 35 days.

Criteria of the mastery of the required skills:

- Learner's answer to an arithmetic or algebraic problem should be an **algorithm** implemented in computer code developed by Learner—which
  - solves *all* problems of the same type;
  - helps to check, analyse, and generalise the solution.
- This applies to so called 'real life' maths problems as well.
- **Proofs** naturally appear as proofs of validity of algorithms.
- Ability to detect own errors is all-important!

The bridge from arithmetic to algebra

The merger of school mathematics with informatics

- Should start simultaneously with transition of mathematical learning from arithmetic to algebra.
- Should be rooted in those aspects of arithmetic which actually belong to computer science but are not usually recognised as such.

Arithmetic already contains

Abstraction The concept of *number* is already a huge abstraction.

Algorithms
- Long division, etc. are algorithms.
- The '**questions method**' allows children to develop their own algorithms which solve many types of arithmetic problems.

Recursion Long division, long multiplication, addition and subtraction of decimals.

Typed variables 'Named numbers' of arithmetic perfectly map to typed variables of computer programming.

Reification Examples were given.

The talk is based on

Borovik, Kondratiev. *A new course 'Algebra + Computer Science': What should be its outcomes and where it should start*, 2023, https://arxiv.org/abs/2212.12257 [math.HO].
Borovik, Kocsis, Kondratiev. *Mathematics and Mathematics Education in the 21st Century*, 2022, arXiv:2201.08364 [math.HO].
Borovik. *Mathematics for makers and mathematics for users*, 2017, http://bit.ly/2qYHtst.
Borovik. *Economy of thought: a neglected principle of mathematics education*, 2017.
Borovik. *Calling a spade a spade: Mathematics in the new pattern of division of labour*, 2016, http://goo.gl/TT6ncO.

Links are clickable.

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For more:

Read *Executive Summary* in

Borovik, Kondratiev. *A new course 'Algebra + Computer Science': What should be its outcomes and where it should start*, 2023, https://arxiv.org/abs/2212.12257 [math.HO].

Thanks!